Indian Statistical Institute

Asymptotic Theory for Some Families of Two-Sample Nonparametric Statistics
Author(s): Lars Holst and J. S. Rao
Source: *Sankhyā: The Indian Journal of Statistics, Series A*, Vol. 42, No. 1/2 (Apr., 1980), pp. 19-52
Published by: Indian Statistical Institute
Stable URL: http://www.jstor.org/stable/25050211
Accessed: 18/06/2010 09:10

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/action/showPublisher?publisherCode=indstatinst.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Indian Statistical Institute is collaborating with JSTOR to digitize, preserve and extend access to *Sankhy: The Indian Journal of Statistics, Series A.*

http://www.jstor.org

Sankhyā : The Indian Journal of Statistics
1980, Volume **42**, Series A, Pts. 1 & 2, pp. 19–52.

ASYMPTOTIC THEORY FOR SOME FAMILIES OF TWO-SAMPLE NONPARAMETRIC STATISTICS

By LARS HOLST*

University of Wisconsin, Madison and Uppsala University, Sweden

and

J. S. RAO

University of California, Santa Barbara, USA

SUMMARY. Let $X_1, \ldots, X_{m-1}$ and $Y_1, \ldots, Y_n$ be independent random samples from two continuous distribution functions F and G respectively on the real line. We wish to test the null hypothesis that these two parent populations are identical. Let $X'_1 \leqslant \ldots \leqslant X'_{m-1}$ be the ordered X-observations. Denote by S_k the number of Y-observations falling in the interval $[X'_{k-1}, X'_k)$, $k = 1, \ldots, m$. This paper studies the asymptotic distribution theory and limiting efficiencies of families of test statistics for the null hypothesis, based on these numbers $\{S_k\}$. Let $h(\,.\,)$ and $\{h_k(\,.\,)\ k = 1, \ldots, m\}$ be real-valued functions satisfying some simple regularity conditions. Asymptotic theory under the null hypothesis as well as under a suitable sequence of alternatives, is studied for test statistics of the form $\sum_{k=1}^{m} h(S_k)$, based symmetrically on S_k's and those of the form $\sum_{k=1}^{m} h_k(S_k)$ which are not symmetric in $\{S_k\}$. It is shown here that tests of the symmetric type have poor asymptotic performance in the sense that they can only distinguish alternatives at a "distance" of $n^{-1/4}$ from the hypothesis. Among this class of symmetric tests, which includes for instance the well known run test and the Dixon test, it is shown that the Dixon test has the maximum asymptotic relative efficiency. On the other hand, tests of the nonsymmetric type can distinguish alternatives converging at the more standard rate of $n^{-1/2}$. Wilcoxon-Mann-Whitney test is an example which belongs to this class. After investigating the asymptotic theory under such alternatives, methods are suggested which allow one to select an "optimal" test against any specific alternative, from among tests of the type $\sum_{k=1}^{m} h_k(S_k)$. Connections with rank tests are briefly explored and some illustrative examples provided.

1. Introduction and notations

Let $X_1, \ldots, X_{m-1}$ and $Y_1, \ldots, Y_n$ be independent random samples from two populations with continuous distribution functions (d.f.s.) $F(x)$ and $G(y)$ respectively. We wish to test if these two populations are identical, i.e., the hypothesis that the two d.f.s. are the same. A simple probability integral transformation carrying $z \to F(z)$ would permit us to assume that the support

*Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

of both the probability distributions is the unit interval $[0, 1]$ and that the first of them is the uniform d.f. on $[0, 1]$. For the purposes of this discussion, this probability transformation can be done without loss of any generality as will be apparent soon. Thus from now on, we will assume that this reduction has been effected and that the first sample is from the uniform distribution $U(0, 1)$. Let $G^* = G \circ F^{-1}$ denote the d.f. of the second sample after the probability transformation. The null hypothesis to be tested, specifies

$$H_0 : G^*(y) = y, \quad 0 \leqslant y \leqslant 1. \qquad \dots \quad (1.1)$$

Let $0 \leqslant X_1' \leqslant \dots \leqslant X_{m-1}' \leqslant 1$ be the order statistics from the first sample. The sample spacings $(D_1, \dots, D_m)$ for the X-values are defined by

$$D_k = D_{km} = X_k' - X_{k-1}', \qquad k = 1, \dots, m \qquad \dots \quad (1.2)$$

where we put $X_0' = 0$ and $X_m' = 1$. Tests based on these sample spacings have been considered in the literature for the goodnesss-of-fit problems. See for instance Darling (1953), Pyke (1965) and Rao and Sethuraman (1975). Define for $k = 1, \dots, m$

$$S_k = \text{number of } y_j\text{'s in the interval } [X_{k-1}', X_k'). \qquad \dots \quad (1.3)$$

Our aim is to study various test statistics based on these numbers $\{S_1, \dots, S_m\}$ for testing H_0. These quantities may be called "spacing-frequencies" (since they denote the frequencies of y's in the sample spacings of the x's) or the "rank-spacings" (since they correspond to the gaps in the ranks of the x's in the combined sample). Since the numbers $\{S_k\}$ as well as the statistics based on them remain invariant under probability transformations, there is no loss of generality in making such a transformation on the data, as was done earlier. It may be remarked here that we take $(m-1)$ instead of the usual m observations in the first sample since this yields m numbers $\{S_1, \dots, S_m\}$ instead of $(m+1)$, leading to slightly simpler notation. Tests based on $\{S_k\}$ have been considered for the two-sample problem in Dixon (1940), Godambe (1961) and Rao (1976).

Our aim is to study the asymptotic theory as m and n tend to infinity. We will do this through a nondecreasing sequence of positive integers $\{m_\nu\}$ and $\{n_\nu\}$ and assume throughout, that as $\nu \to \infty$,

$$m_\nu \to \infty, \; n_\nu \to \infty \text{ and } m_\nu/n_\nu = r_\nu \to \rho, \quad 0 < \rho < \infty. \qquad \dots \quad (1.4)$$

Note that $\{D_k\}$ defined in (1.2) depend on m_ν the number of X-values and it is more appropriate to label them as $\{D_{k\nu}\}$. Similarly the numbers $\{S_k\}$ defined in (1.3) depend on both m_ν and n_ν and should therefore be denoted by $\{S_{k\nu}\}$. Thus we are dealing with triangular arrays of random variables $\{D_{k\nu},\ k = 1, \ldots, m_\nu\}$ and $\{S_{k\nu},\ k = 1, \ldots, m_\nu\}$ for $\nu \geqslant 1$. Corresponding to the ν-th ($\nu \geqslant 1$) array, let $h_\nu(\cdot)$ and $\{h_{k\nu}(\cdot),\ k = 1, \ldots, m_\nu\}$ be real-valued functions satisfying certain regularity conditions (see Condition (A) of Section 2). Define

$$T_\nu = \sum_{k=1}^{m_\nu} h_{k\nu}(S_{k\nu}) \quad \ldots \quad (1.5)$$

and

$$T_\nu^* = \sum_{k=1}^{m_\nu} h_\nu(S_{k\nu}) \quad \ldots \quad (1.6)$$

based on the $(m_\nu - 1)$ X-values and the n_ν Y-values. Though T_ν^* is a special case of T_ν when $\{h_{k\nu}(\cdot)\}$ do not depend on k, we will distinguish these two cases since their asymptotic behaviour is quite different in the non-null situation. It may be noted here that the Wald-Wolfowitz (1940) run test and the Dixon (1940) test are of the form T_ν^* while the Wilcoxon-Mann-Whitney test is of the form T_ν. In fact, any linear function based on the X-ranks in the combined sample, can be expressed as a special case of T_ν. (cf. also Section 5.)

A few words about the notations : Though the quantities m, n, r, D_k, S_k as well as the functions $h(\cdot)$, $\{h_k(\cdot)\}$ depend on ν, for notational convenience the suffix ν is suppressed except where it is essential. Thus for instance, $T_\nu = \sum_{k=1}^{m} h_k(S_k)$, $T_\nu^* = \sum_{k=1}^{m} h(S_k)$ and r will stand for (m/n) etc. The probability law of a random variable (or random vector) X will be denoted by $\mathcal{L}(X)$. A normal distribution with mean μ and covariance matrix Σ will be represented by $N(\mu, \Sigma)$ throughout while $N(0, 0)$ stands for the degenerate distribution at the point zero. For $0 < x < \infty$, $\mathcal{P}(x)$ will represent the Poisson distribution with mean x and

$$\pi_j(x) = e^{-x}.\, x^j/j!, \quad j = 0, 1, 2, \ldots \quad \ldots \quad (1.7)$$

the Poisson probability of j. For $\boldsymbol{p} = (p_1, \ldots, p_m)$, mult $(n, \boldsymbol{p})$ will denote the m-dimensional multinomial distribution with n trials and cell probabilities $(p_1, \ldots, p_m)$. A negative exponential random variable (r.v.) with density

$$e^{-w} \text{ for } w \geqslant 0 \text{ and zero elsewhere} \quad \ldots \quad (1.8)$$

will be denoted throughout by W while $\{W_1, W_2, \ldots\}$ will stand for an independent and identically distributed (i.i.d.) sequence of such r.v.'s. The random variable η will have a geometric distribution with p.d.f.

$$P(\eta = j) = \rho/(1+\rho)^{j+1}, \qquad j = 0, 1, 2, \ldots \qquad \ldots \quad (1.9)$$

for $0 < \rho < \infty$.

The following conditional relationship between these distributions is useful later on. Let W be a negative exponential r.v. as above. Let η denote a r.v. which for given $W = w$, is $\mathcal{P}(w/\rho)$. Then the (unconditional) distribution of η is

$$P(\eta = j) = E\pi_j(W/\rho) = \int_0^\infty (w/\rho)^j \frac{e^{-w/\rho}}{j!} e^{-w}\, dw = \rho/(1+\rho)^{j+1},\ j = 0, 1, 2, \ldots \qquad \ldots \quad (1.10)$$

the same as (1.9) above. Thus η has a geometric distribution if conditional on $W = w$, it has a $\mathcal{P}(w/\rho)$ distribution.

Also for any random variable X_n, we write $X_n = o_p(g(n))$ if $X_n/g(n) \to 0$ in probability and we write $X_n = O_p(g(n))$ if for each $\varepsilon > 0$, there is a $K_\varepsilon < \infty$ such that $P\{|X_n/g(n)| > K_\varepsilon\} < \varepsilon$ for all n sufficiently large. Finally $[x]$ will denote the largest integer contained in x.

We shall consider a sequence of alternatives specified by the d.f.'s

$$G^*_m(y) = y + (L_m(y))/m^\delta, \qquad 0 \leqslant y \leqslant 1 \qquad \ldots \quad (1.11)$$

where $L_m(0) = L_m(1) = 0$ and $\delta \geqslant \frac{1}{4}$. In terms of the original d.f.'s F and G, the null hypothesis specifies $G = F$, while under the alternatives there is a sequence of d.f.s. G_m that converge to F as the sample size increases. Indeed $L_m(\cdot)$ of (1.11) is given by

$$L_m(y) = m^\delta(G_m(F^{-1}(y)) - y). \qquad \ldots \quad (1.12)$$

We assume that there is a function $L(y)$ on (0, 1) to which $L_m(y)$ converges. For further conditions on $L_m(\cdot)$ and $L(\cdot)$ refer to Assumptions (B) and (B*). This sequence of alternatives (1.11) is smooth in a certain sense and has been considered before. See for instance Rao and Sethuraman (1975) or Holst (1972).

The organization of this paper is as follows : In Section 2, some preliminary results are established. Theorem 2.1 gives asymptotic distribution of functions of multinomial frequencies while Theorem 2.2 establishes a result on the limit distributions of non-symmetric spacings statistics, which is of independent interest. These results are combined in Theorem 3.1 to obtain the limit distribution of T_ν under the alternatives (1.11) with $\delta = \frac{1}{2}$. It is clear that putting $L_m(y) \equiv 0$ in this theorem, gives the asymptotic distribution of T_ν under H_0. The problem of finding an asymptotically optimal test for a given sequence of alternatives is considered in Theorem 3.2. Some specific examples are discussed at the end of this section. Section 4 deals with the symmetric statistics T_ν^*. Theorem 4.1 gives the asymptotic distribution of T_ν^* under the sequence of alternatives (1.1) with $\delta = \frac{1}{4}$ while Theorem 4.2 finds the optimal test among the symmetric tests. It is interesting to note that symmetric classes of test statistics T_ν^* can only distinguish alternatives converging to the hypothesis at the slow rate of $n^{-\frac{1}{4}}$ unlike the non-symmetric statistics which can discriminate alternatives converging at the more usual rate of $n^{-\frac{1}{2}}$. Similar results hold for tests based on sample spacings depending on whether or not one considers symmetric statistics. See for instance Rao and Sethuraman (1975) and Rao and Holst (1980). Section 5 contains some further remarks and discussion.

2. Some preliminary results

The following regularity conditions which limit the growth of the functions as well as supply smoothness properties, will be needed for the results of this and the next section.

Condition (A) : The real-valued functions $\{h_k(\cdot)\}$ defined on $\{0, 1, 2, \ldots\}$ satisfy Condition (A) if they are of the form

$$h_k(j) = h(k/(m+1), j), \quad k = 1, \ldots, m \quad j = 0, 1, 2, \ldots \qquad \ldots \quad (2.1)$$

for some function $h(u, j)$ defined for $0 < u < 1$, $j = 0, 1, 2, \ldots$ with the properties

(i) $h(u, j)$ is continuous in u except for finitely many u and the discontinuity set if any, does not depend on j.

(ii) $h(u, j)$ is not of the form $c \cdot j + h(u)$ for some function h on [0, 1] and a real number c.

(iii) For some $\delta > 0$, there exist constants c_1 and c_2 such that

$$|h(u, j)| \leqslant c_1 \cdot [u(1-u)]^{-\frac{1}{2}+\delta} \cdot (j^{c_2}+1)$$
$$\text{for all } 0 < u < 1 \text{ and } j = 0, 1, 2, \ldots \qquad \ldots \quad (2.2)$$

Condition (A′) *:* The real-valued functions $\{g_k(\cdot)\}$ defined on $[0, \infty)$ satisfy Condition (A′) if they are of the form

$$g_k(x) = g(k/(m+1), x), \qquad k = 1, \ldots, m \text{ and } 0 \leqslant x < \infty$$

for some function $g(u, x)$ defined for $0 < u < 1$ and $0 \leqslant x < \infty$ with the properties,

(i) $g(u, x)$ is continuous in u except for finitely many u and the discontinuity set if any, does not depend on x,

(ii) $g(u, x)$ is not of the form $c \cdot x + g(u)$ for some function g on $[0, 1]$ and a real number c, and

(iii) for some $\delta > 0$, there exist constants c_1 and c_2 such that

$$|g(u, x)| \leqslant c_1 \cdot [u(1-u)]^{-\frac{1}{2}+\delta} \cdot (x^{c_2}+1)$$

$$\text{for all } 0 < u < 1 \text{ and } 0 \leqslant x < \infty. \qquad \ldots (2.3)$$

We require the following simple lemma, which is stated without proof.

Lemma 2.1 : *Let $h(u)$ defined for $0 < u < 1$, be continuous except for finitely many u and be bounded in absolute value by an integrable function. Then*

$$(1/m) \sum_{k=1}^{m} h(k/(m+1)) \to \int_0^1 h(u)\, du \quad \text{as} \quad m \to \infty. \quad \square \qquad \ldots (2.4)$$

Turning to the main problem, we will obtain the distribution of T_ν defined in (1.5), essentially in two steps. First we consider the statistic T_ν for given values of the X-spacings $\boldsymbol{D} = \{D_1, \ldots, D_m\}$. Since the numbers $\{S_1, \ldots, S_m\}$ given $\boldsymbol{D}$ have a multinomial distribution, we need a result on the multinomial sums. We formulate this part of the result in Theorem 2.1. The expressions for the asymptotic mean and variance of this conditional distribution of T_ν given $\boldsymbol{D}$, are functions of $\boldsymbol{D}$. In Theorem 2.2, we formulate a general result on the limit distributions of functions of spacings, which allows us to handle in particular, these expressions for the asymptotic mean and variance. Theorem 3.1 of the next section combines these results along with other lemmas given there, thus giving the required asymptotic distribution of T_ν.

It is clear that the conditional distribution of the vector of spacing frequencies $\boldsymbol{S} = (S_1, \ldots, S_m)$ given the spacings vector $\boldsymbol{D} = (D_1, \ldots, D_m)$ is mult $(n, D_1, \ldots, D_m)$. Therefore the test statistic T_ν, conditional on $\boldsymbol{D}$, has under the null hypothesis, the same distribution as the random variable

$$Z_\nu = \sum_{k=1}^{m_\nu} h_k(\varphi_k) \qquad \ldots \quad (2.5)$$

where $(\varphi_1, \ldots, \varphi_m)$ is mult $(n, D_1, \ldots, D_m)$. Since the asymptotic mean and variance of Z_ν can be more simply stated in terms of Poisson random variables, we introduce a triangular array of independent Poisson random variables $\{\xi_{1\nu}, \ldots, \xi_{m_\nu \nu}\}$, $\nu \geqslant 1$ where $\xi_{k\nu}$ is $\mathscr{P}(n_\nu\, p_{k\nu})$ and set

$$\lambda_\nu = \sum_{k=1}^{m} h_k(\xi_{k\nu}), \qquad \ldots \quad (2.6)$$

$$\mu_\nu = E(\lambda_\nu),\ \sigma_\nu^2 = \text{var}\,(\lambda_\nu). \qquad \ldots \quad (2.7)$$

The following theorem on the asymptotic distribution of the multinomial sum Z_ν can be derived as a special case of Theorem 2 of Holst (1979) by taking Poisson r.v.'s $(\xi_{k\nu}, h_k(\xi_{k\nu}))$ in place of $(X_{k\nu}, Y_{k\nu})$ of that theorem.

Theorem 2.1 : *Let* $(\varphi_1, \ldots, \varphi_m)$ *be mult* $(n, p_1, \ldots, p_m)$ *and* Z_ν, μ_ν, *and* σ_ν *be as defined in* (2.5), (2.6) *and* (2.7). *For* $0 < q < 1$, *set* $M = [mq]$ *and*

$$\lambda_{\nu q} = \sum_{k=1}^{M} h_k(\xi_k). \qquad \ldots \quad (2.8)$$

Assume that there exists a $q_0 < 1$ *such that for* $q \geqslant q_0$

$$\sum_{k=1}^{M} p_k \to P_q, \quad 0 < P_q < 1, \qquad \ldots \quad (2.9)$$

and

$$\mathscr{L}\begin{pmatrix} (\lambda_{\nu q} - E\lambda_{\nu q})/m^{\frac{1}{2}} \\ \sum_{1}^{M} (\xi_k - np_k)/n^{\frac{1}{2}} \end{pmatrix} \to N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} A_q & B_q \\ B_q & P_q \end{pmatrix} \right) \qquad \ldots \quad (2.10)$$

where A_q, B_q *and* P_q *are such that as* $q \to 1\text{–}0$,

$$A_q \to A_1,\ B_q \to B_1 \text{ and } P_q \to 1. \qquad \ldots \quad (2.11)$$

Then as $\nu \to \infty$,

$$\mathscr{L}((Z_\nu - \mu_\nu)/\sigma_\nu) \to N(0, A_1 - B_1^2). \quad \square \qquad \ldots \quad (2.12)$$

From (2.6) and (2.7) an explicit expression for the mean is given by

$$\mu_\nu = \sum_{k=1}^{m} \sum_{j=0}^{\infty} h_k(j)\, \pi_j\,(np_k) \qquad \dots \quad (2.13)$$

using the notation (1.7). Under the null hypothesis, we have $p_k = D_k$, $k = 1, \dots, m$ where $\boldsymbol{D}$ are the spacings from $U(0,\ 1)$. Thus we consider

$$\mu(n\boldsymbol{D}) = \mu_\nu(n\boldsymbol{D}) = \sum_{k=1}^{m} \sum_{j=0}^{\infty} h_k(j)\, \pi_j\,(nD_k). \qquad \dots \quad (2.14)$$

This is of the form $\sum_{k=1}^{m} g_k(mD_k)$ where $g_k(x) = \sum_{j=0}^{\infty} h_k(j)\, \pi_j\,(x/r)$.

Statistics based on spacings have been considered earlier by Darling (1953), LeCam (1958), Pyke (1965) and Rao and Sethuraman (1975). Most of these papers, however, discuss the symmetric case, i.e., when $g_k(x) = g(x)$ for all k. As Pyke (1965) pointed out (cf. Section 6.2), LeCam's method could be used to study the more general non-symmetric case. Let $\{g_k(\cdot),\ k = 1, \dots, m\}$ be real-valued measurable functions. For $0 < q \leqslant 1$, let $M_\nu = [m_\nu \cdot q]$. Define

$$G_{q\nu} = \sum_{k=1}^{M} g_k(W_k) \qquad \dots \quad (2.15)$$

where $\{W_1, W_2, \dots\}$ is a sequence of i.i.d. exponential r.v.'s. Then the following theorem states explicitly the asymptotic distribution of statistics of the type (2.14) and is easily established by checking Assumption (6.6) of Pyke (1965).

Theorem 2.2 : *Assume that*

$$0 < var\,(G_{q\nu}) = \sigma^2(G_{q\nu}) < \infty \text{ for all } q \text{ and } \nu, \qquad \dots \quad (2.16)$$

and that for each $q \in (0,\ 1]$

$$\mathscr{L}\begin{pmatrix} (G_{q\nu} - EG_{q\nu})/\sigma(G_{1\nu}) \\ \sum_{1}^{M} (W_k - 1)/m^{\frac{1}{2}} \end{pmatrix} \to N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} A_q & B_q \\ B_q & q \end{pmatrix} \right) \dots \quad (2.17)$$

with A_q *and* B_q *such that*

$$A_q \to A_1 = 1 \text{ as } q \to 1\text{–}0 \qquad \dots \quad (2.18)$$

$$B_q \to B_1 \text{ as } q \to 1\text{–}0. \qquad \dots \quad (2.19)$$

Then, as $\nu \to \infty$,

$$\mathcal{L}\left(\left(\sum_{k=1}^{m} g_k(mD_k)-EG_{1\nu}\right)/\sigma(G_{1\nu})\right) \to N(0,\, 1-B_1^2). \quad \dots \quad (2.20)$$

where $(D_1, \dots, D)$ *are spacings from* $U(0, 1)$. □

The following corollary gives a simple sufficient condition on the functions $g_k(\cdot)$ in order that the above Theorem holds.

Corollary 2.1 : *The asymptotic normality asserted in Theorem* 2.2 *holds for any set of functions* $\{g_k(\cdot)\}$ *which satisfy condition* (A′).

Proof: To prove this corollary, we need to check that the assumptions (2.16) to (2.19) hold when condition (A′) is satisfied. It can be easily checked that if $g(u, x)$ satisfies condition (A′), then

$$\int_0^\infty g(u, x)e^{-x}\, dx,\ \int_0^\infty g^2(u, x)e^{-x}\, dx \text{ as well as } \int_0^\infty g(u, x)(x-1)e^{-x}\, dx$$

satisfy conditions of Lemma 2.1 in u. Thus from the definition of $G_{q\nu}$ and Lemma 2.1, as $m \to \infty$,

$$E(G_{q\nu})/m = (1/m) \sum_{k=1}^{[mq]} Eg(k/(m+1),\, W_k) \to \int_0^q Eg(u, W)\, du \quad \dots \quad (2.21)$$

$$\text{var}\,(G_{q\nu})/m = (1/m) \sum_{k=1}^{[mq]} \text{var}\,(g(k/(m+1), W_k)) \to \int_0^q (\text{var}\, g(u, W))\, du \quad \dots \quad (2.22)$$

and

$$\text{cov}\,(G_{q\nu}, \sum_1^M W_k)/m = (1/m) \sum_{k=1}^{[mq]} \text{cov}\,(g(k/(m+1),\, W_k),\, W_k)$$

$$\to \int_0^q \text{cov}\,(g(u, W),\, W)\, du. \quad \dots \quad (2.23)$$

Again from (2.3) of condition (A′), all these limits are finite. These are also continuous in q so that (2.18) and (2.19) are satisfied.

Finally to check the asymptotic normality in (2.17) or equivalently of

$$\sum_{k=1}^{[mq]} \{a(g_k(W_k)-Eg_k(W_k))+(W_k-1)\} = \sum_{k=1}^{[mq]} g_k^*(W_k), \text{ say} \quad \dots \quad (2.24)$$

for all real a, we have only to verify the Lindeberg condition for the non-identical case. It is easily seen that if $\{g_k(\cdot)\}$ satisfy condition (A′), so do $\{g_k^*(\cdot)\}$ defined in (2.24). Let

$$\sigma_{\hat{k}}^2 = Eg_k^{*^2}(W_k) \text{ and } s_{[mq]}^2 = \sum_{k=1}^{[mq]} \sigma_k^2. \qquad \ldots \quad (2.25)$$

Since $\{g_k^*(\cdot)\}$ satisfy condition (A′), we have as in (2.22) that

$$s_{[mq]}^2/m = (1/m) \sum_{k=1}^{[mq]} \int_0^\infty g_k^{*^2}(w) e^{-w}\, dw \qquad \ldots \quad (2.26)$$

converges to a finite non-zero constant from Lemma 2.1. Now consider

$$(1/s_{[mq]}^2) \sum_{k=1}^{[mq]} \int_{|x| > \varepsilon s_{[mq]}} g_k^{*^2}(x) e^{-x}\, dx$$

$$\leqslant (m/s_{[mq]}^2) \cdot (1/m) \sum_{k=1}^{[mq]} \int_{|x| > \varepsilon s_{[mq]}} c_1[(k/(m+1))(1-k/(m+1)]^{-1+2\delta}$$

$$\cdot (x^{c_2}+1)^2 e^{-x}\, dx$$

$$\leqslant \{mc_1^2/s_{[mq]}^2\}\{(1/m) \sum_{k=1}^{[mq]} [(k/(m+1))(1-k/(m+1))]^{-1+2\delta}\}$$

$$\left\{ \int_{|x| > \epsilon s_{[mq]}} (x^{c_2}+1)^2 e^{-x}\, dx \right\}.$$

As $m \to \infty$, the quantities in the first two parentheses remain bounded because of (2.26) and Lemma 2.1 while the integral in the third parenthesis goes to zero for any $\varepsilon > 0$ since $s_{[mq]}$ is of order $(\sqrt{m})$ from (2.26). Thus the Lindeberg condition is satisfied for (2.24) which proves the assertion. □

3. Asymptotic distribution theory for non-symmetric statistics

We define for later use, the following additional functions

$$g_1(u, x) = \sum_{j=0}^{\infty} h(u, j)\, \pi_j(x), \qquad \ldots \quad (3.1)$$

$$g_2(u, x) = \sum_{j=0}^{\infty} h^2(u, j)\, \pi_j(x) \qquad \ldots \quad (3.2)$$

and

$$g_3(u, x) = \sum_{j=0}^{\infty} h(u, j)(j-x)\, \pi_j(x). \qquad \ldots \quad (3.3)$$

When $h(u, j)$ satisfies condition (A), these functions are well defined for $x > 0$. Further g_1 and g_3 satisfy condition (A′). For instance condition (iii) of (A) implies condition (iii) of (A′) for g_1 since

$$|g_1(u, x)| = \left| \sum_{j=0}^{\infty} h(u, j)\, \pi_j(x) \right|$$

$$\leqslant \sum_{j=0}^{\infty} c_1(u(1-u))^{-\frac{1}{2}+\delta} (1+j^{c_2})\, \pi_j(x)$$

$$\leqslant c_1'(u(1-u))^{-\frac{1}{2}+\delta} (1+x^{c_2'}) \quad \ldots \quad (3.4)$$

using the moments of the Poisson distribution. To see the role of these functions g_1, g_2 and g_3, recall the representation of η given in (1.10). Let E_W, V_W denote the expectation and variance over W while $E_{\eta|W}$ $V_{\eta|W}$ denote the conditional expectation and variance over η given W. Then from the definitions of g_1, g_2, g_3

$$E_\eta h(u, \eta) = E_W E_{\eta|W}\; h(u, \eta) = E_W g_1(u, W/\rho) \quad \ldots \quad (3.5)$$

$$E_\eta h^2(u, \eta) = E_W E_{\eta|W}\; h^2(u, \eta) = E_W g_2(u, W/\rho). \quad \ldots \quad (3.6)$$

And after some elementary calculations,

$$\rho(1+\rho)^{-1} \operatorname{cov}(h(u, \eta), \eta) = E_W E_{\eta|W}\; [h(u, \eta)(\eta - W/\rho)]$$

$$= \operatorname{cov}(g_1(u, W/\rho),\; W) = E_W\; g_3(u, W/\rho). \quad \ldots \quad (3.7)$$

Define

$$\sigma^2 = \int_0^1 \operatorname{var}(h(u, \eta))\, du - \left(\int_0^1 \operatorname{cov}(h(u, \eta), \eta)\, du \right)^2 / \operatorname{var}(\eta). \quad \ldots \quad (3.8)$$

From the Cauchy-Schwartz inequality

$$\left(\int_0^1 \operatorname{cov}(h(u, \eta), \eta)\, du \right)^2 \leqslant \left(\int_0^1 (\operatorname{var} h(u, \eta))^{\frac{1}{2}} (\operatorname{var} \eta)^{\frac{1}{2}}\, du \right)^2$$

$$\leqslant \operatorname{var}(\eta) \cdot \left(\int_0^1 \operatorname{var} h(u, \eta)\, du \right)$$

with equality if and only if $h(u, j) = c \cdot j + h(u)$ for $0 \leqslant u \leqslant 1$ for some real number c and some function $h(u)$. Thus $\sigma^2 > 0$ for any function $h(u, j)$ satisfying condition (A). For $\boldsymbol{x} = (x_1, \ldots, x_m)$, we define

$$\mu(\boldsymbol{x}) = \mu_\nu(\boldsymbol{x}) = \sum_{k=1}^{m_\nu} g_1(k/(m+1),\; x_k) = \sum_{k=1}^{m} \sum_{j=0}^{\infty} h_k(j)\, \pi_j(x_k) \quad \ldots \quad (3.9)$$

and observe that $\mu(n\boldsymbol{D})$ corresponds to the statistic in (2.14).

Before we proceed to state the theorem which gives the asymptotic distribution of T_ν under the alternatives, a few words about the sequence of alternatives. Consider the Y-observations from the distribution function given in (1.11), (1.12), with $\delta = \frac{1}{2}$, i.e.,

$$A_m^{(1)} : G_m^*(y) = G_m(F^{-1}(y)) = y + L_m(y)/m^{\frac{1}{2}}, \quad 0 \leqslant y \leqslant 1. \qquad \dots \quad (3.10)$$

Assumption (B) : For the alternatives in (3.10) with $\delta = \frac{1}{2}$, assume that there exists a continuous function $L(y)$ such that for $0 \leqslant y \leqslant 1$,

$$L_m(y) = m^{\frac{1}{2}}[G_m(F^{-1}(y)) - y] \to L(y) \text{ as } m \to \infty.$$

Also suppose that the derivatives $L_m'(y)$ and $L'(y) = l(y)$ exist and are continuous outside some fixed finite subset in [0, 1] and that finite left and right limits of the derivatives exist on the open interval (0, 1).

Given the X-sample, the probability of a Y-observation falling inside $[X_{k-1}', X_k')$, under the null hypothesis is given by the uniform spacings $\{D_k\}$. On the other hand, under the alternatives (3.10), this probability is given by

$$D_k^* = G_m(F^{-1}(U_k')) - G_m(F^{-1}(U_{k-1}')) = D_k(1 + \Delta_k/m^{\frac{1}{2}}) \qquad \dots \quad (3.11)$$

where U_k', $k = 1, \dots, m$ are order statistics from $U(0, 1)$ with $U_0' = 0$, $U_m' = 1$ and

$$\Delta_k = [L_m(U_k') - L_m(U_{k-1}')]/D_k. \qquad \dots \quad (3.12)$$

Note that $D_k > 0$ with probability one so that Δ_k is a well-defined random variable. We now state the main theorem of this section, whose proof will be completed in Lemmas 3.1 to 3.7. The conditions of this theorem may be slightly weakened but at the expense of added complexity. In any case, the present conditions cover most cases of statistical interest.

Theorem 3.1 : *Let*

$$V_\nu = \sum_{k=1}^{m} (h_k(S_k) - Eh_k(\eta))/m^{\frac{1}{2}} \cdot \sigma \qquad \dots \quad (3.13)$$

where σ *is defined in* (3.8). *In addition to Assumptions* (*A*) *and* (*B*) *assume that for some small* $\varepsilon > 0$,

$$|L_m(t) - L_m(s)| \leqslant c_3(t^\alpha - s^\alpha) \textit{ for } 0 \leqslant s \leqslant t < \varepsilon$$

$$\textit{and for } (1-\varepsilon) \leqslant s \leqslant t \leqslant 1 \qquad \dots \quad (3.14)$$

where $7/8 < \alpha < 1$. *Then under the alternatives* (3.10),

$$\mathcal{L}(V_\nu) \to N(b, 1), \quad \ldots \quad (3.15)$$

where

$$b = \int_0^1 cov\,(h(u, \eta), \eta)\, l\,(u)\, du \cdot \rho/(1+\rho)\, \sigma.$$

Proof: Observe first that the centering constant in (3.13) may be rewritten, using relation (1.10)

$$\sum_{k=1}^{m} Eh_k(\eta) = \sum_{k=1}^{m} \sum_{j=0}^{\infty} h_k(j)\, E\pi_j\,(W/\rho)$$

$$= E\mu_\nu(\boldsymbol{W}/\rho) \quad \ldots \quad (3.16)$$

where $\mu_\nu(\boldsymbol{x})$ is defined in (3.9) and $\boldsymbol{W} = (W_1, \ldots, W_m)$ are i.i.d. exponential r.v.'s. As explained in Section 2, the vector $(S_1, \ldots, S_m)$ given $\boldsymbol{D}^*$ is mult $(n, \boldsymbol{D}^*)$ where the m-vector $\boldsymbol{D}^*$ has the components $\boldsymbol{D}_k^*$ given in (3.11). Using conditional expectations, we may write

$$E(e^{it\sigma V_\nu}) = E\, E(e^{it\sigma V_\nu} \,|\, \boldsymbol{D}^*)$$

$$= E(J_\nu(\boldsymbol{D}^*) K_\nu\,(\boldsymbol{D}^*)) \quad \ldots \quad (3.17)$$

where

$$J_\nu(\boldsymbol{D}^*) = \exp\,(itm^{-\frac{1}{2}}\,[\mu\,(n\boldsymbol{D}^*) - E\mu\,(\boldsymbol{W}/\rho)]) \quad \ldots \quad (3.18)$$

and

$$K_\nu(\boldsymbol{D}^*) = E\Big(\exp\Big(itm^{-\frac{1}{2}}\Big[\sum_{k=1}^{m} h_k(S_k) - \mu(n\,\boldsymbol{D}^*)\Big]\Big)\Big|\boldsymbol{D}^*\Big). \quad \ldots \quad (3.19)$$

Now from Lemma 3.4, it follows that

$$E(J_\nu(\boldsymbol{D}^*)) \to \exp\,(ibt - ct^2/2)$$

with b and c defined in (3.38) and (3.39) respectively. Hence

$$\mathcal{L}(m^{-\frac{1}{2}}\,[\mu_\nu(n\boldsymbol{D}^*) - E\mu(\boldsymbol{W}/\rho)]) \to N(b, c) \quad \ldots \quad (3.20)$$

so that $J_\nu(\boldsymbol{D}^*)$ converges in distribution. By Lemma 3.5, with probability one, i.e., for almost every random vector $\boldsymbol{D}^*$,

$$K_\nu(\boldsymbol{D}^*) \to e^{-dt^2/2} \quad \ldots \quad (3.21)$$

with d as defined in (3.43). Combining (3.20) and (3.21), with probability one, the product $J_\nu(\boldsymbol{D}^*)K_\nu(\boldsymbol{D}^*)$ converges in distribution. But since $|J_\nu(\boldsymbol{D}^*)K_\nu(\boldsymbol{D}^*)| \leqslant 1$, this also implies the convergence of the moments so that

$$E(J_\nu(\boldsymbol{D}^*)K_\nu(\boldsymbol{D}^*)) \to \exp(ibt-(c+d)t^2/2). \quad \ldots \quad (3.22)$$

Using the continuity theorem for characteristic functions and Lemma 3.7, the assertion of the theorem follows. □

Lemma 3.1 : *If the conditions of Theorem* 3.1 *hold, then*

$$m^{-\frac{1}{2}} \sum_{k=1}^{m} \sum_{j=0}^{\infty} h(k/(m+1), j)[\pi_j(nD_k^*)-\pi_j(nD_k)]$$

$$= m^{-1} \sum_{k=1}^{m} \Delta_k \sum_{j=0}^{\infty} h(k/(m+1), j)(j-nD_k)\pi_j(nD_k)+o_p(1) \quad \ldots \quad (3.23)$$

where Δ_k *is as defined in* (3.12).

Proof: Applying the Cauchy-Schwartz inequality on the difference of the two sides in (3.23), we have

$$m^{-\frac{1}{2}} \sum_{k=1}^{m} \sum_{j=0}^{\infty} h_k(j)[\pi_j(nD_k^*)-\pi_j(nD_k)\{1+(j-nD_k)\,\Delta_k/m^{\frac{1}{2}}\}]$$

$$\leqslant m^{-\frac{1}{2}} \sum_{k=1}^{m} \sum_{j=0}^{\infty} |h_k(j)| \cdot |\exp\{j\log(1+\Delta_k/m^{\frac{1}{2}})-nD_k\Delta_k/m^{\frac{1}{2}}$$

$$-1-(j-nD_k)\,\Delta_k/m^{\frac{1}{2}}|\,\pi_j(nD_k)$$

$$\leqslant m^{-\frac{1}{2}} \sum_{k=1}^{m} \Big[\sum_{j=0}^{\infty} h_k^2(j)\pi_j(nD_k)\Big]^{\frac{1}{2}} \Big[\sum_{j=0}^{\infty} |\exp\{j\log(1+\Delta_k/m^{\frac{1}{2}})-nD_k\Delta_k/m^{\frac{1}{2}}\}$$

$$-1-(j-nD_k)\,\Delta_k/m^{\frac{1}{2}}|^2\,\pi_j(nD_k)\Big]^{\frac{1}{2}}. \quad \ldots \quad (3.24)$$

After some elementary calculations, we see that the term in the second square bracket is

$$\exp(D_k\Delta_k^2/r)-1-D_k\Delta_k^2/r. \quad \ldots \quad (3.25)$$

Since $h(u, x)$ satisfies condition (A), using Theorem 2.2 and (3.25), it is clear that the right hand side of (3.24) can be estimated by

$$m^{-\frac{1}{2}} \sum_{k=1}^{m} c_1 \cdot g_2(k/(m+1), nD_k)(1+(mD_k)^{e_2})O_p(D_k \Delta_k^2)$$

$$= \max_k (\Delta_k^2/m^{\frac{1}{2}}) \cdot O_p(1). \qquad \ldots \quad (3.26)$$

Now we show that this $\max (\Delta_k^2/m^{\frac{1}{2}})$ goes to zero in probability when $\alpha > 7/8$. Observe that by (3.14)

$$\begin{aligned} |\Delta_k/m^{\frac{1}{4}}| &= |L_m(U_k')-L_m(U_{k-1}')|/m^{\frac{1}{4}} \cdot D_k \\ &\leqslant (U_k'^{\alpha}-U_{k-1}'^{\alpha})/m^{\frac{1}{4}} \cdot D_k \\ &\leqslant D_k^{\alpha}/m^{\frac{1}{4}} \cdot D_k \qquad \ldots \quad (3.27) \end{aligned}$$

since $(t^{\alpha}-s^{\alpha}) \leqslant (t-s)^{\alpha}$ for $0 < s \leqslant t < 1$ and $\alpha < 1$. Also from Darling (1953) for any $\varepsilon > 0$, we have

$$1/\min_{1 \leqslant k \leqslant m} (m^{2+\varepsilon} D_k) = O_p(1).$$

Therefore from (3.27)

$$\max_{1 \leqslant k \leqslant m} |\Delta_k/m^{\frac{1}{4}}| \leqslant O_p(m^{(2+\varepsilon)(1-\alpha)-\frac{1}{4}}). \qquad \ldots \quad (3.28)$$

Since $\alpha > 7/8$, by choosing $0 < \varepsilon < (4(1-\alpha))^{-1}-2$, $\max |\Delta_k/m^{\frac{1}{4}}| \to 0$ in probability. This proves the lemma. □

Lemma 3.2 : *If the conditions of Theorem* 3.1 *are satisfied then, for any* $\varepsilon > 0$,

$$\limsup_{m \to \infty} |m^{-1} \sum_{k=1}^{[m\varepsilon]} \Delta_k \sum_{j=0}^{\infty} h(k/(m+1), j)(j-nD_k)\pi_j(nD_k)| \leqslant K\varepsilon^{\alpha+\beta} \qquad \ldots \quad (3.29)$$

with probability one.

Proof : In terms of $g_3(u, x)$ defined in (3.4) and which satisfies condition (A′), the expression in (3.29) is

$$m^{-1} \sum_{k=1}^{[m\varepsilon]} \Delta_k g_3(k/(m+1), nD_k)$$

$$= \sum_{k=1}^{[m\varepsilon]} [L_m(U_k')-L_m(U_{k-1}')]g_3(k/(m+1), nD_k)(mD_k)^{-1}. \qquad \ldots \quad (3.30)$$

Using condition (3.14), and writing $M = [m\varepsilon]$, this is

$$0 \leqslant \left| \sum_1^M [L_m(U'_k) - L_m(U'_{k-1})]\, g_3\,(k/(m+1),\, nD_k) \right| (mD_k)^{-1}$$

$$\leqslant c_1 c_3 \sum_1^M (U'^{\alpha}_k - U'^{\alpha}_{k-1})(k/m)^{\beta}[1+(nD_k)^{c_2}]. \qquad \ldots \quad (3.31)$$

We now make use of the representation of the spacings in terms of i.i.d. exponential r.v.'s W_1, $W_2, \ldots$ with mean 1. Writing $\overline{W}_k = \sum_1^k W_j/k$, the RHS in (3.31) is

$$C \cdot \frac{\varepsilon^{\alpha+\beta}}{\overline{W}_m^{\alpha+c_2}} \sum_1^M [\overline{W}_k^{\alpha}(k/M)^{\alpha} - \overline{W}_{k-1}^{\alpha}((k-1)/M)^{\alpha}](k/M)^{\beta} \cdot (W_k^{c_2} + \overline{W}_m^{c_2})$$

$$= C \cdot \varepsilon^{\alpha+\beta} \cdot \overline{W}_m^{-(\alpha+c_2)} \cdot M^{-1} \sum_1^M \overline{W}_k^{\alpha-1}(k/M)^{\alpha+\beta-1}(W_k^{c_2+1} + W_k \cdot \overline{W}_m^{c_2})$$

$$\cdot [\{1-(1-W_k/k \cdot \overline{W}_k)^{\alpha}\}/(W_k/k\overline{W}_k)]. \qquad \ldots \quad (3.32)$$

Now as $k \to \infty$, $\overline{W}_k \to 1$ a.s. and $W_k/k \to 0$ a.s. so that

$$\{1-(1-W_k/k\overline{W}_k)^{\alpha}\}/(W_k/k\overline{W}_k) \to \alpha \quad \text{a.s.} \qquad \ldots \quad (3.33)$$

Using the Hölder inequality,

$$\frac{1}{M} \sum_1^M \overline{W}_k^{\alpha-1}\,(k/M)^{\alpha+\beta-1} W_k^{c_2+1}\,[\{1-(1-W_k/k\overline{W}_k)^{\alpha}\}/(W_k/k\overline{W}_k)]$$

$$\leqslant \left[\frac{1}{M}\sum_1^M \overline{W}_k^{(\alpha-1)p_1}\right]^{1/p_1} \cdot \left[\frac{1}{M}\sum_1^M (k/M)^{(\alpha+\beta-1)p_2}\right]^{1/p_2} \cdot$$

$$\cdot \left[\frac{1}{M}\sum_1^M W_k^{(c_2+1)p_3}\right]^{1/p_3} \cdot \left[\frac{1}{M}\sum_1^M \left\{\frac{1-(1-W_k/k\overline{W}_k)^{\alpha}}{W_k/(k\overline{W}_k)}\right\}^{p_4}\right]^{1/p_4}. \qquad \ldots \quad (3.34)$$

Using the fact that if $a_k \to 1$ as $k \to \infty$, $n^{-1}\sum_1^n a_k \to 1$ as $n \to \infty$, the RHS in (3.34) converges a.s. to the finite limit

$$1 \cdot \left[\int_0^1 u^{(\alpha+\beta-1)p_2}\, du\right]^{1/p_2} \cdot \left[E W^{(c_2+1)p_3}\right]^{1/p_3} \cdot \alpha.$$

Similarly the other term involving $W_k \cdot \overline{W}_m^{e_2}$ in (3.32) can be handled so that we get the desired result. □

Lemma 3.3 : *Under the conditions of Theorem* 3.1,

$$m^{-1} \sum_{k=1}^{m} \Delta_k \, g_3(k/(m+1), nD_k)$$

$$\to \int_0^1 l(u) \, cov\,(h(u, \eta), \eta) \, du \cdot (\rho/1+\rho) \text{ in probability.} \quad \ldots \quad (3.35)$$

Proof : For any fixed $\varepsilon > 0$, we may consider the sum in (3.35) as consisting of 3 parts viz., $\sum_{k=1}^{[m\varepsilon]}$, $\sum_{k=[m\varepsilon]}^{[m(1-\varepsilon)]}$ and $\sum_{k=[m(1-\varepsilon)]}^{m}$. Lemma 3.2 shows that the first sum is negligible. A similar analysis can be used to demonstrate that the third term is also bounded a.s. by $K\varepsilon^{\alpha+\beta}$. In view of (3.7), it is enough to show that

$$m^{-1} \sum_{k=[m\varepsilon]}^{[m(1-\varepsilon)]} \Delta_k g_3(k/(m+1), nD_k) \to \int_{\varepsilon}^{1-\varepsilon} l(u) \cdot E(g_3(u, W/\rho) \, du \quad \ldots \quad (3.36)$$

in probability. The proof will then be complete since ε is arbitrary.

By our assumption $L'_m(y) = l_m(y)$ exists and is continuous except possibly for a finite number of points on $(\varepsilon, 1-\varepsilon)$. If $l_m(y)$ is continuous, then boundedness of $l_m(y)$ along with the fact $g_3(u, x)$ satisfies condition (A′) allows us to apply Theorem 2.2 as follows : from the Glivenko-Cantelli theorem,

$$\max_k \; |\Delta_k - l_m(k/m+1)| \to 0 \text{ with probability } 1.$$

Also from Theorem 2.2,

$$m^{-1} \sum_{k=[m\varepsilon]}^{[m(1-\varepsilon)]} g_3(k/(m+1), \; nD_k) = O_p(1).$$

Hence the sum in (3.36) has the same probability limit as

$$m^{-1} \sum_{k=[m\varepsilon]}^{[m(1-\varepsilon)]} l_m(k/(m+1)) \cdot g_3(k/(m+1), nD_k)$$

which from Theoren 2.2 is the required limit given in (3.36).

Now if $l_m(\cdot)$ has a finite set of discontinuity points inside $(\varepsilon, 1-\varepsilon)$, this will not create any problem since the function is bounded in this interval.

Suppose that $l_m(y)$ is continuous in $(0, 1)$ except at $y = y_0$. By our assumptions $l_m(y)$ has finite left and right limits at this point and the point does not depend on m. Take $\delta > 0$ so that $0 < y_0 - \delta < y_0 + \delta < 1$. From our assumptions and the Glivenko-Cantelli theorem it follows that with probability one, $|\Delta_k|$ is bounded whenever $|k/m - y_0| < \delta$ and m is sufficiently large. From this it is easily seen by analogous arguments that the contribution to the sum (3.36) from such terms in the neighborhood of y_0 can be made arbitrarily small by choosing δ sufficiently small. It is obvious that the situation of a finite set of discontinuities of the first kind can be handled the same way, if the discontinuity set does not depend on m. This completes the proof of Lemma 3.3. □

Lemma 3.4 : *Let*

$$J_\nu(\boldsymbol{D}^*) = exp\,(itm^{-\frac{1}{2}}\,[\mu_\nu(n\,\boldsymbol{D}^*) - E\mu(\boldsymbol{W}/\rho)])$$

be as defined in (3.18). *Then under the conditions of Theorem* 3.1,

$$E(J_\nu(\boldsymbol{D}^*)) \to exp\,(ibt - ct^2/2) \qquad \ldots \quad (3.37)$$

where

$$b = \int_0^1 cov\,(h(u, \eta), \eta)\, l(u) du\ \rho/(1+\rho) \qquad \ldots \quad (3.38)$$

and

$$c = \int_0^1 var\, g_1(u, W/\rho) du - \left(\int_0^1 cov\,(W, g_1(u, W/\rho)) du \right)^2. \qquad \ldots \quad (3.39)$$

Proof : We can write

$$J_\nu(\boldsymbol{D}^*) = \exp\,(itm^{-\frac{1}{2}}([\mu_\nu(n\boldsymbol{D}^*) - \mu_\nu(n\boldsymbol{D})] + [\mu_\nu(n\boldsymbol{D}) - E\mu_\nu(W/\rho)]). \quad \ldots \quad (3.40)$$

In Lemmas 3.1 to 3.3, we already established that the first part $m^{-\frac{1}{2}}[\mu_\nu(n\boldsymbol{D}^*) - \mu_\nu(n\boldsymbol{D})]$ converges in probability to b. Thus we need only show that

$$E(\exp\,(itm^{-\frac{1}{2}}[\mu_\nu(n\boldsymbol{D}) - E\mu_\nu(W/\rho)])) \to \exp\,(-ct^2/2). \qquad \ldots \quad (3.41)$$

Since $g_1(u, x)$ satisfies condition (A′), Corollary 2.1 of Section 2 holds and the asymptotic normality of

$$\mu_\nu(n\boldsymbol{D}) = \sum_{k=1}^{m_\nu} g_1(k/(m+1),\ nD_k)$$

is assured by Theorem 2.2. Further var $(g_1(u, W/\rho))$ and cov $(W, g_1(u, W/\rho))$ as functions in u, satisfy the conditions of Lemma 2.1, so that as $\nu \to \infty$

$$m^{-1} \operatorname{var} \left(\sum_{k=1}^{m} g_1(k/(m+1), W_k/\rho) \right) - m^{-2} \operatorname{cov}^2 \left(\sum_{1}^{m} W_k, \sum_{1}^{m} g(k/(m+1), W_k/\rho) \right)$$

$$\to \int_0^1 \operatorname{var}(g_1(u, W/\rho))du - \left(\int_0^1 \operatorname{cov}(W, g_1(u, W/\rho))du \right)^2$$

$$= c. \quad \square \qquad \ldots \quad (3.42)$$

Lemma 3.5 : *Under the assumptions of Theorem* 3.1, *with probability one, i.e., for almost every* $\boldsymbol{D}^*$

$$K_\nu(\boldsymbol{D}^*) = E\left(exp \left(itm^{-\frac{1}{2}} \left[\sum_{1}^{m} h_k(S_k) - \mu(n\boldsymbol{D}^*) \right] \right) \Big| \boldsymbol{D}^* \right)$$

$$\to exp\,(-dt^2/2)$$

where

$$d = \int_0^1 E[g_2(u, W/\rho) - g_1(u, W/\rho)]^2 du - \rho \left(\int_0^1 Eg_3(u, W/\rho)du \right)^2. \qquad \ldots \quad (3.43)$$

Proof : The lemma will be proved by verifying that the conditions of Theorem 2.1 hold and showing $d = A_1 - B_1^2$. First we have by the Glivenko-Cantelli theorem that with probability one

$$\sum_{k=1}^{M} D_k^* = U'_M + m^{-\frac{1}{2}} L_m(U'_M) \to q = P_q \qquad \ldots \quad (3.44)$$

where $M = [mq]$ and U'_k is the k-th order statistic from $U(0, 1)$. Clearly since $P_q = q \to 1$ as $q \to 1-$, conditions (2.9) and part of (2.11) of Theorem 2.1 hold. For real numbers a and b, consider

$$h_1(u, j) = ah(u, j) + bj. \qquad \ldots \quad (3.45)$$

It is easy to verify that if $h(u, j)$ satisfies condition (A), then so does $h_1(u, j)$. Consider

$$\zeta'_q = m^{-\frac{1}{2}} \sum_{k=1}^{M} (h_1(k/(m+1), \xi_k) - Eh_1(k/(m+1), \xi_k)) \qquad \ldots \quad (3.46)$$

where $\xi_1, \ldots, \xi_m$ are independent and ξ_k is $\mathcal{P}(nD_k^*)$. From the assumptions, it follows that for some positive constants $c_1, c_2, \ldots$ we have

$$V(\zeta_q') = m^{-1} \sum_{k=1}^{M} \operatorname{var}(h_1(k/(m+1), \xi_k))$$

$$\leqslant m^{-1}c_1 \sum_{k=1}^{M} [(k/(m+1))(1-k/(m+1))]^{\beta}((nD_k^*)^{c_2}+1)$$

$$\leqslant m^{-1}c_1' \sum_{k=1}^{m} [(k/(m+1))(1-k/(m+1))]^{\beta}(nD_k^*)^{c_2}+c_3$$

$$\leqslant c_1'' \left(\sum_{k=1}^{m} (nD_k^*)^{c_4}/m)^{c_5}\right)+c_3 \qquad \ldots \quad (3.47)$$

by the Hölder inequality and Lemma 2.1. From the assumption (3.14),

$$nD_k^* = nD_k+n(L_m(U_k')-L_m(U_{k-1}'))/m^{-\frac{1}{2}}$$

$$\leqslant nD_k+K_1D_k\, m^{\frac{1}{2}}+K_2\, D_k^{\alpha}\, m^{\frac{1}{2}}$$

$$\leqslant K_3(mD_k)+K_2(mD_k)^{\alpha}m^{\frac{1}{2}-\alpha}. \qquad \ldots \quad (3.48)$$

Using the representation $mD_k = (W_k/\overline{W}_m)$, it follows by the strong law of large numbers that for $c > 0$,

$$\lim_{m\to\infty} m^{-1} \sum_{1}^{m} (mD_k)^c$$

is finite with probability one. As $\alpha > \frac{1}{2}$, we have, using the binomial theorem

$$m^{-1} \sum_{1}^{m} (mD_k^*)^{c_6} \leqslant m^{-1}K \sum_{1}^{m} (K_3mD_k+K_2(mD_k)^{\alpha}m^{\frac{1}{2}-\alpha}+1)^{c_6+1} \to K_4$$

with probability one. Thus with probability one

$$\limsup \operatorname{var}(\zeta_q') < \infty. \qquad \ldots \quad (3.49)$$

Now we will verify that

$$\liminf \operatorname{var}(\zeta_q') > 0. \qquad \ldots \quad (3.50)$$

By assumption (A), it follows that there exists an interval $[a, b] \subset (0, 1)$ and integers $j_1 \neq j_2$ such that $h_1(u, j_1) \neq h_1(u, j_2)$ for $a \leqslant u \leqslant b$. Again from the strong law of large numbers and our assumptions, it is easily seen that for any $0 < C < D < \infty$, with probability one

$$\# \{k: a < k/(m+1) < b, \quad C < nD_k^* < D\}/m \to K_1 > 0.$$

Therefore for n sufficiently large,

$$\text{var}(\zeta_q') \geqslant \sum_{a(m+1) < k < b(m+1)} \text{var}(h_1(k/(m+1), \xi_k))/m \geqslant K_2 > 0$$

with probability one. Hence (3.50) is satisfied with probability one. In a similar fashion it follows that

$$\limsup \sum_{k=1}^{m} E\,|\,h_1(k/(m+1), \xi_k)\,|^{2+\varepsilon}/m < \infty.$$

Therefore the Liapunov condition

$$\sum_{1}^{M} E\,|\,h_1(k/(m+1), \xi_k)\,|^{2+\varepsilon}/(m \text{ var}(\zeta_q'))^{1+\varepsilon/2}$$

$$= m^{-\varepsilon/2} \sum_{1}^{M} E\,|\,h_1(k/(m+1), \xi_k)\,|^{2+\varepsilon}/(\text{var}(\zeta_q'))^{1+\varepsilon/2} \cdot m$$

$$\to 0 \text{ as } m \to \infty, \qquad \dots \quad (3.51)$$

is satisfied with probability one. Thus

$$\mathscr{L}(\zeta_q'/(\text{var}(\zeta_q'))^{\frac{1}{2}}) \to N(0,\ 1)$$

with probability one. By the next Lemma 3.6, we have that in probability

$$\text{var}(\zeta_q') \to (a^2A_q + 2abB_q\rho^{-\frac{1}{2}} + b^2q\rho^{-1})$$

where $A_q \to A_1$, $B_q \to B_1$ as $q \to 1-$. This verifies that the assumptions of Theorem 2.1 are satisfied with probability one. From the definition (3.43) of d as well as the expressions (3.54) and (3.55) for A_q and B_q, it follows

$$d = A_1 - B_1^2 \qquad \dots \quad (3.52)$$

which proves the lemma. □

Lemma 3.6 : *Given* $\boldsymbol{D}^*$, *let* $(\xi_1, \dots, \xi_m)$ *be independent and* ξ_k *be* $\mathscr{P}(n\boldsymbol{D}_k^*)$. *Under the assumptions of Theorem* 3.1

$$m^{-1} \sum_{1}^{m} \text{var}(h_1(k/(m+1), \xi_k)) \to a^2A_q + 2abB_q\rho^{-\frac{1}{2}} + b^2q\rho^{-1} \qquad \dots \quad (3.53)$$

in probability where

$$A_q = \int_0^q E[g_2(u, W/\rho) - g_1(u, W/\rho)]^2 \, du \qquad \ldots \quad (3.54)$$

and

$$B_q = \rho^{-\frac{1}{2}} \int_0^q E(g_3(u, W/\rho)) \, du. \qquad \ldots \quad (3.55)$$

Proof : Recall from (3.45) that $h_1(u, j) = ah(u, j) + bj$. By calculations similar to those in Lemma 3.1, it follows that, for instance

$$m^{-1} \sum_{k=1}^{M} \sum_{j=0}^{\infty} h^2(k/(m+1), j)[\pi_j(nD_k^*) - \pi_j(nD_k)] \to 0$$

in probability. Using Theorem 2.2, we get

$$m^{-1} \sum_{k=1}^{M} h^2(k/(m+1), j)\pi_j(nD_k) \to \int_0^q (Eg_2(u, W/\rho)) \, du$$

in probability. Therefore

$$m^{-1} \sum_{1}^{M} Eh^2(k/(m+1), \xi_k) \to \int_0^q (Eg_2(u, W/\rho)) \, du.$$

The other terms can be handled analogously which proves the assertion. □

Lemma 3.7 :

$$c + d = \sigma^2 \qquad \ldots \quad (3.56)$$

where c, d, σ^2 *are defined in* (3.39), (3.43) *and* (3.8) *respectively.*

Proof : From the definitions (3.39) and (3.43) of c and d and from identities (3.5), (3.6) and (3.7), we get

$$c + d = \int_0^1 \operatorname{var}(g_1(u, W/\rho)) \, du - \left(\int_0^1 \operatorname{cov}(W, g_1(u, W/\rho)) \, du \right)^2$$

$$+ \int_0^1 E[g_2(u, W/\rho) - g_1(u, W/\rho)]^2 \, du - \rho \left(\int_0^1 Eg_3(u, W/\rho) \, du \right)^2$$

$$= \int_0^1 V_W(E_{\eta \mid W}(h(u, \eta))) \, du + \int_0^1 E_W(V_{\eta \mid W}(h(u, \eta))) \, du$$

$$-(1+\rho) \left[\int_0^1 \operatorname{cov}(h(u, \eta), \eta) \, \rho/(1+\rho) \, du \right]^2$$

$$= \int_0^1 \operatorname{var}(h(u, \eta)) \, du - \left[\int_0^1 \operatorname{cov}(h(u, \eta), \eta) \, du \right]^2 \rho^2/(1+\rho)$$

$$= \sigma^2. \; \square \qquad \ldots \quad (3.57)$$

These lemmas 3.1 to 3.7 complete the proof of Theorem 3.1. The following lemma gives a simple sufficient condition for (3.14) to hold.

Lemma 3.8 : *A sufficient condition for* (3.14) *to hold in a neighborhood of the origin is that*

$$0 \leqslant L'_m(u) \leqslant c \cdot u^{\alpha-1} \text{ for } 0 < u < \varepsilon. \qquad \dots \quad (3.58)$$

Proof : We have for $0 < s < t < \varepsilon$

$$0 \leqslant \int_s^t (cu^{\alpha-1} - L'_m(u))\, du = c(t^\alpha - s^\alpha)/\alpha - (L_m(t) - L_m(s)).$$

Since $L_m(0) = 0$ and $L'_m(u) \geqslant 0$, the assertion follows. □

Corollary 3.1 : *Under the null hypothesis* (1.1), *the asymptotic distribution of* V_ν *defined in* (3.13) *is* $N(0, 1)$. □

This result is a direct consequence of Theorem 3.1 and is obtained by taking $l(u) \equiv 0$, $0 < u < 1$ in (3.15). This corollary regarding the null distribution of V_ν can also be reformulated in the following interesting form using Lemma 2.1.

Corollary 3.1′ : *Let* $\eta_1, \eta_2, \dots$ *be a sequence of i.i.d. geometric random variables with p.d.f. given in* (1.9). *Then the asymptotic null distribution of* $\sum_1^m h_k(S_k)$ *is* $N\left(E\left(\sum_1^m h_k(\eta_k)\right), \text{var}\left(\sum_1^m h_k(\eta_k) - \beta \sum_1^m \eta_k\right)\right)$ *where* β *is the regression coefficient given by*

$$\beta = \text{cov}\left(\sum_1^m h_k(\eta_k), \sum_1^m \eta_k\right) \Big/ \text{var}\left(\sum_1^m \eta_k\right). \ \square$$

See also Holst (1979), example 2. We now consider the problem of finding the optimal choice of the function $h(u, j)$ for a given alternative sequence (3.10), i.e., a given sequence of functions $L_m(u)$ with the property

$$L_m(u) = m^{\frac{1}{2}}(G_m(F^{-1}(u)) - u) \to L(u) \text{ as } m \to \infty. \qquad \dots \quad (3.59)$$

Theorem 3.2 : *If the sequence of alternatives is such that the assumptions of Theorem* 3.1 *are fulfilled, then an asymptotically most powerful* (*AMP*) *test of the hypothesis against the simple alternative* (3.10) *is to reject* H_0 *when*

$$\sum_{k=1}^m l(k/m+1) S_k > c \qquad \dots \quad (3.60)$$

where l is the derivative of L, mentioned in (3.59). *The asymptotic distribution of this optimal statistic is given by*

$$\mathcal{L}\left(m^{-\frac{1}{2}} \sum_{k=1}^{m} l(k/(m+1))(S_k-1/\rho)\right) \to N(0,\ \sigma^2) \qquad \ldots \quad (3.61)$$

under H_0 *with*

$$\sigma^2 = \left(\int_0^1 l^2(u)\ du\right)(1+\rho)/\rho^2 \qquad \ldots \quad (3.62)$$

while under the alternatives (3.10) *satisfying* (3.59)

$$\mathcal{L}\left(m^{-\frac{1}{2}} \sum_{k=1}^{m} l(k/(m+1))(S_k-1/\rho)\right) \to N\left(\rho^{-1} \int_0^1 l^2(u)\ du,\ \sigma^2\right), \qquad \ldots \quad (3.63)$$

Proof : From Theorem 3.1, it follows that the asymptotic power of the test which rejects H_0 when $\sum_{k=1}^{m} h(k/(m+1),\ S_k) > c$ is determined by

$$P_h = \int_0^1 \operatorname{cov}(h(u,\eta),\eta)\ l(u)\ du \Bigg/ \Bigg[\int_0^1 (\operatorname{var}\ h(u,\eta))\ du - \left\{ \int_0^1 \operatorname{cov}(h(u,\eta),\eta)\ du \right\}^2 \Big/ \operatorname{var}(\eta) \Bigg]^{\frac{1}{2}}. \qquad \ldots \quad (3.64)$$

Using the same argument as in Lemma 3.1 of Holst (1972), we have that this quantity is maximized when

$$h(u, j) = l(u) \cdot j. \qquad \ldots \quad (3.65)$$

The results on the asymptotic distributions follow directly from Theorem 3.1 and Corollary 3.1 for the above special case. □

From this result, it follows that the AMP test of level α is explicitly given by : Reject H_0 if

$$\left[\sum_{k=1}^{m} l(k/(m+1)(S_k-1/\rho) \right] \Big/ \left[m \left(\int_0^1 l^2(u)\ du \right)(1+\rho)\rho^{-2} \right]^{\frac{1}{2}} > \lambda_\alpha \qquad \ldots \quad (3.66)$$

where λ_α is the upper α-percentile of the $N(0,\ 1)$ distribution. Also from Theorem 3.2 we find that the asymptotic power of this test in terms of the standard normal c.d.f. is given by the expression

$$\Phi\left(-\lambda_\alpha + \left(\int_0^1 l^2(u)\ du/(1+\rho) \right)^{\frac{1}{2}}\right). \qquad \ldots \quad (3.67)$$

Furthermore it is easily seen from Theorem 3.1 that the Pitman Asymptotic Relative Efficiency (ARE) in using $h(u, j) = d(u) \cdot j$ for some function d on $[0, 1]$ instead of the optimal $h(u, j) = l(u) \cdot j$ is

$$e = \left(\int_0^1 d(u)l(u)du \right)^2 \Big/ \left\{ \int_0^1 d^2(u)du - \left(\int_0^1 d(u)du \right)^2 \right\} \left\{ \int_0^1 l^2(u)du \right\} \quad \ldots \quad (3.68)$$

3.A. *Example : translation alternatives.* We now consider some applications of the above results on non-symmetric tests. First we shall look at the translation alternatives. Let $X_1, \ldots, X_{m-1}$ be absolutely continuous i.i.d. random variables with distribution F. Let $Y_1, \ldots, Y_n$ be i.i.d. with d.f. G. We wish to test

$$H_0 : G(x) = F(x)$$

against the sequence of translation alternatives

$$A_m^{(1)} : G(x) = G_m(x) = F(x-\theta/m^{\frac{1}{2}}). \quad \ldots \quad (3.69)$$

Let $f(\mathrm{x}) = F'(x)$ be continuous. Then as $m \to \infty$

$$L_m(u) = m^{\frac{1}{2}}[G_m(F^{-1}(u))-u] \to -\theta f(F^{-1}(u)) = L(u), \text{ say.} \quad \ldots \quad (3.70)$$

And if $f'(x)$ exists and is continuous except for at most finitely many x's then, at the continuity points of $f'(F^{-1}(u))$ we have

$$l_m(u) \to l(u) = -\theta f'(F^{-1}(u))/f(F^{-1}(u)). \quad \ldots \quad (3.71)$$

We now illustrate how Theorem 3.2 may be used to obtain the asymptotically optimal test statistic based on $\{S_k\}$.

Example: (A van der Waerden or normal score type test) : For the normal d.f.

$$F(x) = \Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp\left(-\tfrac{1}{2} t^2\right) dt, \quad -\infty < x < \infty$$

we find

$$-f'(F^{-1}(u))/f(F^{-1}(u)) = \Phi^{-1}(u).$$

It is easy to check that the required regularity conditions of Theorem 3.1 are satisfied. Hence the AMP test is based on the statistic

$$T = \sum_{k=1}^{m} \Phi^{-1}(k/(m+1))S_k \quad \ldots \quad (3.72)$$

from Theorem 3.2. Using the facts

$$\int_0^1 (\Phi^{-1}(u))^2 du = \int_{-\infty}^{\infty} x^2\varphi(x)dx = 1 \text{ and } \sum_{k=1}^{m} \Phi^{-1}(k/(m+1)) = 0,$$

we have under the null hypothesis that

$$(T/m^{\frac{1}{2}}) \to N(0, (1+\rho)/\rho^2). \quad \ldots \quad (3.73)$$

From Theorem 3.2, the asymptotic power for a one-sided test of level α is $\Phi(-\lambda_\alpha+\theta(1+\rho)^{-\frac{1}{2}})$, the same as that of the Student's t-test.

To find the Pitman efficiency of the Wilcoxon test relative to the optimal test based on (3.72), we only need to calculate (3.68). Since

$$\int_0^1 (2u-1)\Phi^{-1}(u)du = \int_{-\infty}^{\infty} 2(\Phi(x)-1)x\,\varphi(x)\,dx = 2\int_{-\infty}^{\infty} (\varphi(x))^2\,dx = \pi^{-\frac{1}{2}}$$

and $\int_0^1 (2u-1)^2\,du = 1/3$, $\int_0^1 (\Phi^{-1}(u))^2\,du = 1$, using formula (3.68) the ARE of Wilcoxon test versus the normal scores type test

$$e = 3/\pi. \quad \ldots \quad (3.74)$$

The test statistic (3.72) has the same asymptotic properties as the Fisher-Yates-Terry-Hoeffding and van der Waerden's rank tests.

3.B *Example : scale alternatives.* Next we consider absolutely continuous positive random variables under scale alternatives. Let $X_1, \ldots, X_{m-1}$ be i.i.d. $F(x)$ and $Y_1, \ldots, Y_n$ be i.i.d. $G(y)$ with $F(0) = G(0) = 0$. We wish to test

$$H_0 : G(x) = F(x), \quad \ldots \quad (3.75)$$

against the scale alternatives

$$H_1^{(m)} : G(x) = G_m(x) = F(x(1+\theta/m^{\frac{1}{2}})). \quad \ldots \quad (3.76)$$

If the density $f(x) = F'(x)$ is continuous, then as $m \to \infty$,

$$L_m(u) = m^{\frac{1}{2}}(G_m(F^{-1}(u))-u) \to L(u) = -\theta f(F^{-1}(u)) \cdot F^{-1}(u). \quad \ldots \quad (3.77)$$

And if $f'(x)$ exists and is continuous except for finitely many points, then analogous to (3.71),

$$l_m(u) \to l(u) = -\theta[1+f'(F^{-1}(u)) \cdot F^{-1}(u)/f(F^{-1}(u))] \qquad \ldots \quad (3.78)$$

where f' exists. Optimal statistics based on $\{S_k\}$ can be derived just as in the case of translation alternatives.

Example : (Savage or exponential score test). For the exponential distribution $F(x) = (1-e^{-x})$ for $x > 0$ we find

$$l(u) = -\theta(1+\log(1-u)). \qquad \ldots \quad (3.79)$$

The assumptions of Theorem 3.1 can be verified and hence an optimal statistic is given from Theorem 3.2, by

$$T = \sum_{k=1}^{m} \log(1-k/(m+1))(S_k-1/\rho). \qquad \ldots \quad (3.80)$$

Since $\int_0^1 (1+\log(1-u))^2 \, du = 1$ we get that

$$\mathcal{L}(T/m^{\frac{1}{2}}) \to N(0, (1+\rho)/\rho^2) \qquad \ldots \quad (3.81)$$

and that the asymptotic power is

$$\Phi(-\lambda_\alpha+\theta(1+\rho)^{-\frac{1}{2}}). \qquad \ldots \quad (3.82)$$

The ARE of Wilcoxon statistic relative to T in (3.80) above is 3/4. The statistic T is an approximation to the Savage statistic (see Lehmann, 1975 p. 103). The UMP test for the above situation is the test based on $\sum_{k=1}^{m-1} X_k / \sum_{k=1}^{n} Y_k$ which has the same asymptotic power (3.82) as the statistic T in (3.80).

4. Asymptotic distribution theory for symmetric statistics

This section deals with the class of statistics symmetric in $\{S_1, \ldots, S_m\}$, i.e., statistics of the form

$$T_\nu^* = \sum_{k=1}^{m_\nu} h(S_k) \qquad \ldots \quad (4.1)$$

for some given function $h(j)$. Such symmetric tests are an important subclass of the rotationally invariant tests and hence are suited for testing the equality of two circular populations. Clearly this class of symmetric statistics is also covered by the asymptotic theory discussed in the last section. Indeed if the function $h_k(j)$ does not vary with k, i.e., the function $h(u, j)$ of the last section is a function only of j and is independent of u, then we obtain the symmetry in the numbers $\{S_1, \ldots, S_m\}$. But since $\int_0^1 l(u)\, du = 0$, it follows from Theorem 3.1 and Corollary 3.1 that the asymptotic distribution of T_v^* under the sequence of alternatives (3.10) coincides with that under the null hypothesis. Thus symmetric statistics of the type (4.1) cannot distinguish alternatives that are at a 'distance' of $n^{-\frac{1}{2}}$ and have power zero against such close alternatives. Therefore in order to make efficiency comparisons, we have to consider the more distant alternatives with $\delta = 1/4$ in (1.11). Let

$$A_m^{(2)} : G_m^*(y) = y + L_m(y)/m^{1/4}, \quad 0 \leqslant y \leqslant 1$$

with

$$L_m(u) = m^{1/4}(G_m(F^{-1}(u)) - u). \qquad \ldots \quad (4.2)$$

For this symmetric situation, we will make the following slightly stronger assumptions :

Assumption (B*) : Assume L_m is twice differentiable on [0, 1] and there is a function $L(u)$, $0 \leqslant u \leqslant 1$, which is twice continuously differentiable and such that

$$L(0) = L(1) = 0, \ \sup_{0 \leqslant u \leqslant 1} |L_m''(u) - L''(u)| = o(1) \qquad \ldots \quad (4.3)$$

where $L' = l$ and $L'' = l'$ denote the first and second derivatives of L. Notice that for such smooth alternatives, the following also hold :

$$\sup_{0 \leqslant u \leqslant 1} |L_m(u) - L(u)| = o(1), \ \sup_{0 \leqslant u \leqslant 1} |L_m'(u) - l(u)| = o(1). \qquad \ldots \quad (4.4)$$

We define analogous to (3.11) and (3.12)

$$D_k^* = D_k(1 + \Delta_k^*/m^{1/4}) \text{ with } \Delta_k^* = [L_m(U_k') - L_m(U_{k-1}')]/D_k. \qquad \ldots \quad (4.5)$$

We observe that under the above regularity conditions, we have

$$\max_{1 \leqslant k \leqslant m} |\Delta_k^*| \leqslant \sup_{0 \leqslant u \leqslant 1} |l_m(u)| \leqslant K < \infty. \qquad \ldots \quad (4.6)$$

The following theorem gives the asymptotic distribution of the symmetric statistics T_ν^* under the alternatives (4.2).

Theorem 4.1 : *Suppose that there exist constants c_1 and c_2 such that*

$$|h(j)| \leqslant c_1(j^{c_2}+1) \text{ for all } j. \quad \dots \quad (4.7)$$

Let $L_m(u)$ satisfy Assumption(B*) *and let*

$$V_\nu^* = \sum_{k=1}^{m} (h(S_k)-Eh(\eta))/m^{\frac{1}{2}}\sigma \quad \dots \quad (4.8)$$

where

$$\sigma^2 = var\,(h(\eta))-[cov\,(h(\eta),\ \eta)]^2/var\,(\eta) \quad \dots \quad (4.9)$$

and η is the geometric random variable defined in (1.9). *Then under the alternatives* (4.2)

$$\mathcal{L}(V_\nu^*) \to N(A,\ 1) \text{ as } \nu \to \infty \quad \dots \quad (4.10)$$

where

$$A = \left(\int_0^1 l^2(u)du\right) cov\,(h(\eta),\ \eta(\eta-1)-4\eta/\rho)\rho^2/2(1+\rho)^2\sigma. \quad \dots \quad (4.11)$$

Proof : Following the method used in the proof of Theorem 3.1, it suffices to show that

$$m^{-\frac{1}{2}}[\mu_\nu(n\boldsymbol{D}^*)-\mu_\nu(n\boldsymbol{D})] \to A \text{ in probability.}$$

We have

$$m^{-\frac{1}{2}}[\mu_\nu(n\boldsymbol{D}^*)-\mu_\nu(n\boldsymbol{D})]$$

$$= m^{-\frac{1}{2}} \sum_{k=1}^{m} \sum_{j=0}^{\infty} h(j)[\pi_j(n\,D_k^*)-\pi_j(nD_k)]$$

$$= m^{-\frac{1}{2}} \sum_{k=1}^{m} \sum_{j=0}^{\infty} h(j)\pi_j(nD_k)[(1+\Delta_k/m^{1/4})^j \exp(-nD_k\Delta_k/m^{1/4})$$

$$-1-j-nD_k\,\Delta_k/m^{1/4}-\{j(j-1)-2jnD_k+(nD_k)^2\}\Delta_k^2/m^{\frac{1}{2}}$$

$$+m^{-3/4} \sum_{k=1}^{m} \sum_{j=0}^{\infty} h(j)\,\pi_j(nD_k)(j-nD_k)\,\Delta_k$$

$$+m^{-1} \sum_{k=1}^{m} \sum_{j=0}^{\infty} h(j)\pi_j(nD_k)\{j(j-1)-2jnD_k+(nD_k)^2\}\,\Delta_k^2/2. \quad \dots \quad (4.12)$$

After some direct but tedious calculations, it may be verified that the probability limits of the first two terms on the RHS of (4.12) are zero while that of the third term is

$$\left(\int_0^1 l^2(u)du \right) \cdot (\mathrm{cov}\,(h(\eta), \eta(\eta-1)-4\eta/\rho) \cdot \rho^2/2(1+\rho)^2)$$

under the assumptions, thus completing the proof. □

Taking $l(u) \equiv 0$, $0 < u < 1$ in Theorem 4.1 or putting $h_k(j) = h(j) \; \forall \; k$ in Corollaries 3.1, 3.1′, we get the following result on the asymptotic null distribution for the symmetric statistics.

Corollary 4.1 : *Let* V_v^* *be as defined in* (4.8). *Then under the null hypothesis* (1.1) V_v^* *has asymptotically a* $N(0, 1)$ *distribution if the function* $h(\cdot)$ *satisfies condition* (4.7).

As in Theorem 3.2, we now consider a result on the optimal choice of the function $h(\cdot)$ for the symmetric case.

Theorem 4.2 : *For the sequence of alternatives given by* (4.2), *satisfying the conditions of Theorem* 4.1, *the asymptotically most powerful* (*AMP*) *test is of the form : Reject* H_0 *when*

$$\sum_{k=1}^{m} S_k(S_k-1) > c. \qquad \ldots \quad (4.13)$$

Proof : From Theorem 4.1, it follows that the asymptotic power of a test of the form (4.1) is a maximum when the quantity A given in (4.11) is maximized. Observe that

$$\mathrm{cov}\,(\eta, \eta(\eta-1)-4\eta/\rho) = 0 \qquad \ldots \quad (4.14)$$

and

$$\mathrm{var}\,(h(\eta))-\mathrm{cov}^2\,(h(\eta), \eta)/\,\mathrm{var}\,(\eta) = \mathrm{var}\,(h(\eta)-\beta\eta) \qquad \ldots \quad (4.15)$$

where β is the usual linear regression coefficient

$$\beta = \mathrm{cov}\,(h(\eta), \eta)/\mathrm{var}\,(\eta). \qquad \ldots \quad (4.16)$$

Therefore we can rewrite

$$\begin{aligned} 2\rho^{-2}(1+\rho)^2 A/\int_0^1 l^2(u)du &= \mathrm{cov}\,(h(\eta)-\beta\eta, \eta(\eta-1)-4\eta/\rho)/[\mathrm{var}\,(h(\eta)-\beta\eta)]^{\frac{1}{2}} \\ &= \mathrm{cor}\,(h(\eta)-\beta\eta, \eta(\eta-1)-4\eta/\rho) \cdot [\mathrm{var}\,(\eta(\eta-1)-4\eta/\rho)]^{\frac{1}{2}} \\ &\leqslant [\mathrm{var}\,(\eta(\eta-1)-4\eta/\rho)]^{\frac{1}{2}} = 2\rho^{-2}(1+\rho) \qquad \ldots \quad (4.17) \end{aligned}$$

with equality in (4.17) if and only if

$$h(\eta)-\beta\eta = a[\eta(\eta-1)-4\eta/\rho]+b$$

for some real numbers a and b. Thus A is maximized by $h(\eta) = \eta(\eta-1)$ and

$$\max_h A = \int_0^1 l^2(u)\, du/(1+\rho). \quad \square \qquad \ldots \quad (4.18)$$

Using Theorem 4.1, we further have that under H_0,

$$\mathcal{L}\left(\left[\sum_{k=1}^{m} S_k(S_k-1)-2m/\rho^2\right] \Big/ m^{\frac{1}{2}}\,[2\rho^{-2}(1+\rho)]\right) \to N(0, 1) \quad \ldots \quad (4.19)$$

and that the asymptotic power for a test of level α is

$$\Phi\left[-\lambda_\alpha+\left(\int_0^1 l^2(u)du\right)\Big/(1+\rho)\right].$$

Further, from the above proof we see that the ARE in using $\Sigma h(S_k)$ instead of $\Sigma\, S_k(S_k-1)$ is

$$e = \text{cor}^2\,(h(\eta)-\beta\eta,\ \eta(\eta-1)-4\eta/\rho). \qquad \ldots \quad (4.20)$$

The statistic $\sum_{k=1}^{m} S_k^2$ which is equivalent to $\sum_{k=1}^{m} S_k(S_k-1)$ was proposed by Dixon (1940). Blumenthal (1963) and Rao (1976) discuss the ARE of this test while Blum and Weiss (1957) consider the consistency properties. Blum and Weiss (1957) also show that the Dixon test is asymptotically LMP against "linear" alternatives with density $\{1+c(y-\frac{1}{2})\}$, $0 \leqslant y \leqslant 1$ ($|c| \leqslant 2$) but we have shown that Dixon test is indeed AMP against alternatives of the form (4.2).

For a nonnegative integer r, if we define

$$h(x) = \begin{cases} 1 & \text{for} \quad x = r \\ 0 & \text{otherwise,} \end{cases} \qquad \ldots \quad (4.21)$$

then

$$T_v^* = \sum_{k=1}^{m} h(S_k)$$

is the statistic $Q_m(r)$, the proporiton of values among $\{S_k\}$ which are equal to r. This statistic has been discussed in Blum and Weiss (1957) from the point of consistency. Our results establish the asymptotic normality of $Q_m(r)$ under H_0 as well as under the sequence of alternatives (4.2). After some computations we find from Corollary 4.1 that under the null hypothesis

$$\mathcal{L}(m^{\frac{1}{2}}[Q_m(r)-\rho/(1+\rho)^{r+1}]) \to N(0, \sigma^2) \quad \dots \quad (4.22)$$

where

$$\sigma^2 = \{\rho/(1+\rho)^{r+1}\}[1-(\rho/(1+\rho)^{r+1})\{1+(r-1)/\rho)^2(\rho^2/(1+\rho)\}]. \quad \dots \quad (4.23)$$

The Wald-Wolfowitz run test (1940) is related to $Q_m(0)$. Let U be the number of runs of X's and Y's in the combined sample. The hypothesis H_0 is rejected when U/m is too small. From the definition of $Q_m(r)$, it follows easily that

$$|(U/m)-2(n/m)(1-Q_m(0))| \leqslant 1/m. \quad \dots \quad (4.24)$$

Thus the asymptotic distribution of U/m is the same as that of $2\rho(1-Q_m(0))$ and we thus have, under H_0,

$$\mathcal{L}(m^{\frac{1}{2}}[(U/m)-2/(1+\rho)]) \to N(0, 4\rho/(1+\rho)^3). \quad \dots \quad (4.25)$$

Therefore the ARE of the run-statistic against the Dixon's statistic is $\rho/(1+\rho)$ as shown in Rao (1976).

5. Further remarks and discussion

It is interesting to note that the theory developed in this paper gives tests based on $\{S_k\}$ which are asymptotically equivalent to the corresponding rank tests in all the known examples discussed in Section 3. For a unified approach to the theory of rank tests see Chernoff and Savage (1958) or Hajek and Sidak (1967). We conjecture that in general, given any rank test, one can construct a test of the form (3.60) which has asymptotically the same null distribution and power. If this is the case, then the theory presented here seems to lead to much simpler test statistics which are linear in $\{S_k\}$ as compared to the corresponding optimal rank tests based on score functions. Using the fact that tests *linear* in $\{S_k\}$ are linear in the ranks $\{R_k\}$, one can derive the asymptotic distributions of statistics of the form (3.60) from rank theory. But neither the more general results of Theorem 3.1 nor the fact that tests such as (3.60) are asymptotically optimal, seems to follow from rank theory. Further relationships between these two groups of tests is under investigation. It may also be remarked that the theory presented here, especially Theorems 4.1 and 4.2, covers many other tests that are not based on ranks as, for instance, the run test and the median test and seems to be more general to that extent.

The theorems presented here can also be applied to study similar test statistics when the samples are censored. For instance, suppose that the samples are censored at the right by $X'_{[(m-1)q]}$, the $[(m-1)q]$-th order statistic in the X-sample. Under the same assumptions as in Theorem 3.2, we obtain in the same way that optimal test statistic is given by

$$T = \sum_{k=1}^{[(m-1)q]} l(k/(m+1))(S_k - 1/\rho). \qquad \dots \quad (5.1)$$

Under H_0,

$$\mathcal{L}(T/m^{\frac{1}{2}}) \to N(0, \left[\int_0^q l^2(u)du - \left(\int_0^q l(u)du\right)^2\right](\rho+1)/\rho^2)$$

and the asymptotic power is

$$\Phi(-\lambda_\alpha + \left(\int_0^q l^2(u)du \middle/ \left\{\left[\int_0^q l^2(u)du - \left(\int_0^q l(u)du\right)^2\right](1+\rho)\right\}^{1/2}\right)). \qquad \dots \quad (5.2)$$

For results on censoring in rank theory see, for instance, Rao, Savage and Sobel (1960) or Johnson and Mehrotra (1972).

Acknowledgement. The authors wish to thank the referee whose comments led to many improvements in the presentation of this paper.

References

Blum, J. R. and Weiss, L. (1957) : Consistency of certain two-sample tests. *Ann. Math. Statist.*, **28**, 242-246.

Blumenthal, S. (1963) : The asymptotic normality of two test statistics associated with the two-sample problem. *Ann. Math. Statist.*, **34**, 1513-1523.

Chernoff, H. and Savage, I. R. (1958) : Asymptotic normality and efficiency of certain nonparametric statistics. *Ann. Math. Statist.*, **29**, 972-994.

Darling, D. A. (1953) : On a class of problems related to the random division of an interval. *Ann. Math. Statist.*, **24**, 239-253.

Dixon, W. J. (1940) : A criterion for testing the hypothesis that two samples are from the same population. *Ann. Math. Statist.*, **11**, 199-204.

Godambe, V. P. (1961) : On the two-sample problem : A heuristic method for constructing tests. *Ann. Math. Statist.*, **32**, 1091-1107.

Hájek, J. and Šidák, Z. (1967) : *Theory of Rank Tests*, Academic Press, New York.

Holst, L. (1972) : Asymptotic normality and efficiency for certain goodness of fit tests. *Biometrika*, **59**, 137-145.

——— (1979) : Two conditional limit theorems with applications, *Ann. Statist.*, **7**, 551-559.

Johnson, R. A. and Mehrotra, K. G. (1972) : Locally most powerful rank tests for the two-sample problem with censored data, *Ann. Math. Statist.*, **43**, 823-831.

Le Cam, L. (1958) : Un théorems sur la division d'un intervalle par des points pris au hasard. *Pub. Inst. Statist. Univ. Paris*, **7**, 7-16.

Lehmann, E. L. (1975) : *Nonparametric Statistical Methods Based on Ranks*, Holden-Dey, Inc., San Francisco.

Pyke, R. (1965) : Spacings. *Jour. Roy. Statist. Soc.*, Ser. B, **27**, 395-449.

Rao, J. S. (1976) : Some tests based on arc lengths for the circle. *Sankhyā*, Ser. B, **38**, 329-338.

Rao, J. S. and Holst, L. (1980) : Some results on the asymptotic spacings theory. Submitted for publication.

Rao, J. S. and Sethuraman, J. (1975) : Weak convergence of empirical distribution functions of random variables subject to perturbations and scale factors. *Ann. Statist.*, **3**, 299-313.

Rao, U. V. R., Savage, I. R. and Sobel, M. (1960) : Contributions to the theory of rank order statistics : The two sample censored case. *Ann. Math. Statist.*, **31**, 415-426.

Wald, A. and Wolfowitz, J. (1940) : On a test whether two samples are from the same population. *Ann. Math. Statist.*, **11**, 147-162.

Paper received : January, 1979.

Revised : August, 1979.